Five Year Integrated M.Sc. Examination, 2022-2023
Semester - VII
Code: PH-4-7-1

Time: Four Hours | Subject: Advanced Statistical Mechanics | Full Marks: 80

Questions are of values as indicated in the margin
Answer question number **one** and any **four** from the rest

1. Answer any **four** questions:

$4 \times 5 = 20$

   (a) Explain why does the sound waves propagating through a solid can't have an arbitrarily high frequency. Briefly discuss the limitations of Debye's model.

   (b) Briefly explain the concept of Chandrasekhar limit in case of white dwarf star.

   (c) Calculate the partition function of two Bosons each of which can occupy any of the two energy levels of 0, ϵ and 2ϵ. Calculate the partition function if there are two Fermions instead Bosons.

   (d) Calculate the dispersion relation of the sound waves propagating through a monatomic chain.

   (e) Using diagrammatic method calculate energies and degeneracies of ground state, first excited state and second excited state of 2D Ising chain at low temperature limit.

   (f) Qualitatively explain the absence of magnetic ordered phase in 1D Ising chain at finite temperature by using the minimum free energy argument.

2. (a) Calculate the density of states of photons and hence calculate the partition function of the Photon gas.

   (b) Using the partition function derive Planck's distribution law and hence prove the Wien's displacement law.

   (c) Calculate the total energy of the photon gas. Without doing explicit calculation, show that the energy density of a photon gas varies as the fourth power of absolute temperature (T^4).

(2+3)+(2+3)+(3+2)=15

3. (a) Calculate the total number of single phonon states of a solid with N particle occupying V volume. Hence obtain an expression for Debye frequency (ω_D).

   (b) Write the grand canonical partition function for Fermion gas and calculate the average occupation number ($n(\epsilon_p)$) of the p-th level with energy ϵ_p. Show that the total number of Fermions in a system (N) can be expressed as $N = \sum_p n(\epsilon_p)$.

   (c) Consider an electron moving in a constant magnetic field pointing in the $\hat{z}$ direction $\vec{B} = (0, 0, B)$. Choosing an appropriate gauge, explain the existence of Landau levels.

(3+1)+5+6=15

4. (a) Calculate the average energy (E) and average pressure (p) of a non-relativistic Bose gas. Hence prove that the equation of state for the such Bose gas is $pV = \frac{2}{3}E$.

   (b) Calculate the average number of particles (N) of a Bose gas at low temperature limit. Show that below the critical temperature T_c, the number of particle occupies the ground state is given by

$$\frac{n_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2} .$$

   Using $g_n(1) = \zeta(n)$, obtain an expression for T_c. Here all notations have their usual meaning.

   (c) Briefly explain the Cosmic Microwave Background Radiation.

(3+3+1)+(2+3+1)+2=15

5. (a) Prove the following relation for Fermi gas

$$\frac{n\lambda^3}{g_s} = f_{3/2}(z) \,,$$

   where n is the particle density, g_s is the spin degeneracy of states, λ is the thermal wavelength ($\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$), z is the fugacity ($z = e^{\mu\beta}$) and $f_n(z)$ is the Fermi function defined as

$$f_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x + 1} dx \,.$$

(b) Assuming Sommerfeld expansion in low temperature limit, i.e.

$$\frac{n\lambda^3}{g_s} = f_{3/2}(z) = \frac{(\ln z)^{3/2}}{\Gamma(5/2)}\left[1 + \frac{\pi^2}{6}\frac{3}{4}(\ln z)^{-2} + \dots\right] >> 1\,,$$

prove the following relation

$$\ln z = \beta\mathcal{E}_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\mathcal{E}_F}\right)^2 + \dots\right]\,,$$

where $\mathcal{E}_F$ is the Fermi energy defined as

$$\lim_{T\to 0}\ln z = \beta\mathcal{E}_F\,.$$

(c) What is the value of the chemical potential of photon gas? Explain your answer briefly.

6+6+3=15

6. (a) Define magnetization(m). Using mean field theory show that the energy of Ising chain in presence of external magnetic field B can be expressed as

$$E_{mf} = \frac{1}{2}Nqm^2 - (Jqm + B)\sum_i s_i\,,$$

where J is the strength of spin-spin interaction, q in number of nearest neighbours, N is the total number of spins and s_i is the spin at i-th lattice site.

(b) Hence calculate the partition function and show that

$$m = \tanh(\beta B + \beta Jqm)\,.$$

(c) Graphically solve the above equation for $B = 0$ case and comments on the possibility of ferro-para transition below critical temperature (T_c).

(d) Show that zero field magnetization varies as $m_0 \sim \pm(T_c - T)^{1/2}$ near critical point and with field magnetization varies as $m \sim B^{1/3}$.

(2+2)+3+3+5=15